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absolute term be zero, follows the principle here employed, that for $f(z)$ to vanish for $z = \infty$ it is necessary and sufficient that the coefficient of its highest power be evanescent.

NOTE 2. We have found that a pair of roots of $f(x) = 0$ may be expressed by $y_p+z_p(\alpha_p\pm\beta_p\sqrt{-1})$, where we shall suppose $\alpha_p\pm\beta_p\sqrt{-1}$ to denote the values of $\sqrt[n]{-1}$ given by the formula

$$\cos\frac{(2\lambda+1)\pi}{n}\pm\sin\frac{(2\lambda+1)\pi}{n}\sqrt{-1},$$

where $\lambda=p$, p being any integer from 0 to n inclusive. A pair is also expressed by $y_q+z_q(\alpha_q\pm\beta_q\sqrt{-1})$, where $\alpha_q\pm\beta_q\sqrt{-1}$ denotes the values of $\sqrt[n]{-1}$ given by the above trigonometrical formula when $\lambda = q$, q being any integer, except p, from 0 to n. The question arises: does $y_q+z_q(\alpha_q\pm\beta_q\sqrt{-1})$ necessarily represent a different pair of roots from that represented by $y_p+z_p(\alpha_p\pm\beta_p\sqrt{-1})$, by different here meaning not equal. This conclusion seems necessary; since, otherwise, there would be more than one expression for some pairs of roots, and none at all for others, there being but $\frac{1}{2}n$ pairs of values of $\sqrt[n]{-1}$. If we can assume this result we are led to an interesting generalization respecting the roots of all equations of, at least, even degree, which could then be expressed by the formula

$$x = y + z\left[\cos\frac{(2\lambda+1)\pi}{n}\pm\sin\frac{(2\lambda+1)\pi}{n}\sqrt{-1}\right].$$

In this when $y = 0$, $z = \sqrt[n]{A}$, we have the well known formula for the roots of $x^n+A=0$, when n is even; for $z=0$, the formula represents real roots; if $z_p\beta_p = z_q\beta_q$ and $y_p+z_p\alpha_p = y_q+z_q\alpha_q$, there are two pairs of equal imaginary roots, or one pair of equal real roots if $z=0$. The subject seems worthy of further investigation.

A COLLECTION OF PROOFS OF THE RELATION
$r'+r''+r'''-r = 4R$.

BY MARCUS BAKER, U. S. COAST SURVEY, WASHINGTON, D. C.

IF R is the radius of the circle circumscribed about a plane triangle, r the radius of the inscribed circle, and if r', r'', r''' be the radii of the escribed circles, then is the above relation true. The earliest appearance of this relation in print that I have seen is in J. H. Van Swinden's Elemente der geometrie von C. F. A. Jacobi: Jena, 1834; though the relation was doubtless found earlier than this. The collection here made is not the result of a special effort to get all the proofs known of it but rather to put together such as have been met with incidentally in mathematical reading.

The relation in the sixth proof

$$R+r=x+y+z$$

is older than the relation discussed in this article and was first given by Carnot in his Geometry of position, the german translation of which by Schumacher appeared in 1808. This incidentally answers in part the query by the editor at p. 122, Vol. IV, of the ANALYST. For, though Carnot does not prove that

$$2(R+r)=a\cot A+b\cot B+c\cot C$$

he does prove, and by the same method as the editor of the ANALYST, page 123, Vol. IV, that

$$R+r=x+y+z.$$

FIRST PROOF.—$K=sr=abc\div 4R=2R^2\sin A\sin B\sin C$,

$$\therefore\quad \frac{r}{R}=\frac{2R\sin A.\sin B.\sin C}{\frac{1}{2}(a+b+c)};$$

and since $a=2R\sin A$, $b=2R\sin B$, $c=2R\sin C$

$$\frac{r}{R}=2\frac{\sin A.\sin B.\sin C}{\sin A+\sin B+\sin C}=\frac{\sin A.\sin B.\sin C}{2\cos\frac{1}{2}A.\cos\frac{1}{2}B.\cos\frac{1}{2}C}$$
$$=4\sin\tfrac{1}{2}A.\sin\tfrac{1}{2}B.\sin\tfrac{1}{2}C; \qquad (1)$$

$$\frac{r'}{R}=2\frac{\sin A.\sin B.\sin C}{-\sin A+\sin B+\sin C}=\frac{\sin A.\sin B.\sin C}{2\cos\frac{1}{2}A.\sin\frac{1}{2}B.\sin\frac{1}{2}C}$$
$$=4\sin\tfrac{1}{2}A.\cos\tfrac{1}{2}B.\cos\tfrac{1}{2}C; \qquad (2)$$

$$\frac{r''}{R}=2\frac{\sin A.\sin B.\sin C}{\sin A-\sin B+\sin C}=\frac{\sin A.\sin B.\sin C}{2\sin\frac{1}{2}A.\cos\frac{1}{2}B.\sin\frac{1}{2}C}$$
$$=4\cos\tfrac{1}{2}A.\sin\tfrac{1}{2}B.\cos\tfrac{1}{2}C; \qquad (3)$$

$$\frac{r'''}{R}=2\frac{\sin A.\sin B.\sin C}{\sin A+\sin B-\sin C}=\frac{\sin A.\sin B.\sin C}{2\sin\frac{1}{2}A.\sin\frac{1}{2}B.\cos\frac{1}{2}C}$$
$$=4\cos\tfrac{1}{2}A.\cos\tfrac{1}{2}B.\sin\tfrac{1}{2}C. \qquad (4)$$

But $\sin\frac{1}{2}A.\cos\frac{1}{2}B.\cos\frac{1}{2}C+\cos\frac{1}{2}A.\sin\frac{1}{2}B.\cos\frac{1}{2}C+\cos\frac{1}{2}A.\cos\frac{1}{2}B.\sin\frac{1}{2}C$
$-\sin\frac{1}{2}A.\sin\frac{1}{2}B\sin\frac{1}{2}C=1$; (5)

therefore subtracting (1) from the sum of (2), (3) and (4) and clearing of fractions, we have

$$r'+r''+r'''-r=4R.$$

SECOND PROOF.—Because $r'=s\tan\frac{1}{2}A$, $r''=s\tan\frac{1}{2}B$, $r'''=s\tan\frac{1}{2}C$ and $r=s\tan\frac{1}{2}A\tan\frac{1}{2}B\tan\frac{1}{2}C$;

$\therefore\ r'+r''+r'''-r=s(\tan\frac{1}{2}A+\tan\frac{1}{2}B+\tan\frac{1}{2}C-\tan\frac{1}{2}A\tan\frac{1}{2}B\tan\frac{1}{2}C)$,

$$=\frac{s}{\cos\frac{1}{2}A\cos\frac{1}{2}B\cos\frac{1}{2}C}, \quad \text{[Chauv's Trig. p. 39 (177).]}$$

$$=4R. \quad \text{[Chauv's Trig. p. 76 (286).]}$$

Third Proof.—Let O', O'', O''' represent the centres of the escribed circles, then is $O''\,O''' = 2R'\sin\frac{1}{2}(B+C) = 4R\cos\frac{1}{2}A$;

$$O''\,O''' = (r''+r''')\sec\tfrac{1}{2}A;$$

therefore
$$r''+r''' = 4R\cos^2\tfrac{1}{2}A; \tag{6}$$
similarly
$$r'''+r' = 4R\cos^2\tfrac{1}{2}B; \tag{7}$$
$$r'+r'' = 4R\cos^2\tfrac{1}{2}C. \tag{8}$$

Adding
$$\begin{aligned} r'+r''+r''' &= 2R(\cos^2\tfrac{1}{2}A+\cos^2\tfrac{1}{2}B+\cos^2\tfrac{1}{2}C),\\ &= R(3+\cos A+\cos B+\cos C),\\ &= 4R(1+\sin\tfrac{1}{2}A\sin\tfrac{1}{2}B\sin\tfrac{1}{2}C).\end{aligned}$$

But $r = 4R\sin\frac{1}{2}A\sin\frac{1}{2}B\sin\frac{1}{2}C$, by (1), and therefore
$$r'+r''+r'''-r = 4R.$$

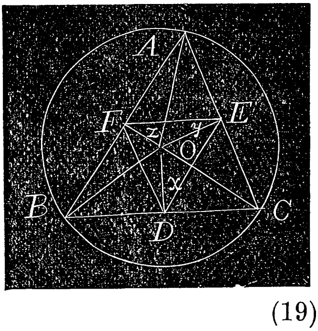
Fourth Proof.—The circle which circumscribes the triangle ABC bisects OO' and therefore

$$\tfrac{1}{2}(r'-r) = R-R\cos A,\ \text{or}\ r'-r = 4R\sin^2\tfrac{1}{2}A; \tag{9}$$
similarly
$$r''-r = 4R\sin^2\tfrac{1}{2}B, \tag{10}$$
$$r'''-r = 4R\sin^2\tfrac{1}{2}C. \tag{11}$$

Adding
$$\begin{aligned} r'+r''+r'''-3r &= 4R(\sin^2\tfrac{1}{2}A+\sin^2\tfrac{1}{2}B+\sin^2\tfrac{1}{2}C),\\ &= 2R(3-\cos A-\cos B-\cos C),\\ &= 2R(2-4\sin\tfrac{1}{2}A.\sin\tfrac{1}{2}B.\sin\tfrac{1}{2}C)\\ &= 2R[2-(r\div R)],\\ &= 4R-2r;\\ r'+r''+r'''-r &= 4R.\end{aligned}$$

Fifth Proof.—From the *third* and *fourth* proofs we have
$$r''+r''' = 4R\cos^2\tfrac{1}{2}A, \tag{6}$$
$$r'-r = 4R\sin^2\tfrac{1}{2}A. \tag{9}$$
Adding
$$r'+r''+r'''-r = 4R.$$

Sixth Proof.—By Ptolemy's Theorem
$$cy+bz = aR, \tag{12}$$
$$az+cx = bR, \tag{13}$$
$$bx+ay = cR, \tag{14}$$
and
$$ax+by+cz = (a+b+c)r. \tag{15}$$
Adding and rejecting the factor $a+b+c$,
$$r+R = x+y+z. \tag{16}$$
Again
$$ax+by+cz = (-a+b+c)r' \tag{17}$$
$$= (a-b+c)r'' \tag{18}$$
$$= (a+b-c)r''' \tag{19}$$



Replacing (15) with (17), (18) and (19) respectively and subtracting from the sum of (12), (13) and (14) we obtain
$r'-R=-x+y+z$ (20), $r''-R=x-y+z$ (21), $r'''-R=x+y-z$, (22).
Snbt. (16) from the sum of (20), (21) and (22), $r'+r''+r'''-r=4R$.

SEVENTH PROOF.—

$$R = \frac{\frac{1}{2}a}{\sin A} = \frac{\frac{1}{2}b}{\sin B} = \frac{\frac{1}{2}c}{\sin C},$$

$$= \frac{\frac{1}{2}a\cot A}{\cos A} = \frac{\frac{1}{2}b\cot B}{\cos B} = \frac{\frac{1}{2}c\cot C}{\cos C},$$

$$= \frac{\frac{1}{2}(a\cot A + b\cot B + c\cot C)}{\cos A + \cos B + \cos C},$$

$$= \frac{\frac{1}{2}(a\cot A + b\cot B + c\cot C)}{1 + 4\sin\frac{1}{2}A\sin\frac{1}{2}B\sin\frac{1}{2}C},$$

$$= \frac{x+y+z}{1+(r\div R)}.$$

$\therefore\ r+R = x+y+z.$ Similarly we obtain the rest and thence $r'+r''+r'''-r = 4R.$

EIGHTH PROOF.—[See Chauvenet's Ttig., 7th ed., p.78 (298).]

$$x = R\cos A,\ y = R\cos B,\ z = R\cos C;$$

hence $r+R = x+y+z = R(\cos A + \cos B + \cos C),$

or $$\frac{r}{R} = \cos A + \cos B + \cos C - 1. \qquad (23)$$

Similarly $$\frac{r'}{R} = -\cos A + \cos B + \cos C + 1, \qquad (24)$$

$$\frac{r''}{R} = \cos A - \cos B + \cos C + 1, \qquad (25)$$

$$\frac{r'''}{R} = \cos A + \cos B - \cos C + 1. \qquad (26)$$

Subtracting (23) from the sum of (24), (25) and (26) we obtain

$$r'+r''+r'''-r = 4R.$$

NINTH PROOF.—$K = sr = (s-a)r' = (s-b)r'' = (s-c)r''';$

$$\therefore\quad r'+r''+r'''-r = K\left(\frac{1}{s-a} + \frac{1}{s-b} + \frac{1}{s-c} - \frac{1}{s}\right)$$

$$= K\left(\frac{abc}{s(s-a)(s-b)(s-c)}\right),$$

$$= \frac{abc}{K} = 4R.$$

TENTH PROOF.—$(s-a)r' + (s-b)r'' + (s-c)r''' = 3sr. \qquad (27)$

$$ar'+br''+cr'''+2K = 2K',$$

or $$ar'+br''+cr''' = 4Rs - 2rs. \qquad (28)$$

Adding (27) and (28)

$$s(r'+r''+r''') = 4Rs + rs,$$

or $$r'+r''+r'''-r = 4R.$$

Eleventh Proof. — The circle which circumscribes the triangle ABC is the nine point circle of the triangle $O'O''O'''$ and consequently bisects $O''O'''$ and also OO'. (O denoting the intersection of the lines AO', BO'', CO'''.) Because M is the middle point of $O''O'''$

$$MP = \tfrac{1}{2}(r''+r'''),$$

and because N is the middle point of OO'

$$NP = \tfrac{1}{2}(r' - r).$$

Adding $$MP+NP = 2R = \tfrac{1}{2}(r'+r''+r'''-r),$$
$$\therefore\ 4R = r'+r''+r'''-r.$$

A PROBLEM WITH SOLUTION.

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Let $u', u'', \ldots, v', v'', \ldots$ be such functions of x that each vanishes when $x = a$; and also the derivatives of each, up to (but not including) those of $s', s'', \ldots, t', t'', \ldots$ order, respectively. It is then required to find the limit as x approaches a, of $\frac{u'u''\ldots}{v'v''\ldots}$.

We proceed to find the order and value of the first non-vanishing derivative of each product.

Since differentiation always produces expressions which are homogeneous with respect to the introduced $\frac{d}{dx}$, the $s'+s''+\ldots.$ derivative of $u'u''\ldots.$ contains the term $A\left(\frac{d}{da}\right)^{s'}u'\left(\frac{d}{dx}\right)^{s''}u''\ldots.$ which does not vanish; and besides, other terms of equal order, cannot be without a vanishing derivative, and are all zero, unless some other derivative of not greater than order $s'+s''+\ldots.$ is infinite. This indeterminate case we will exclude; and then all terms of less order also vanish.

We find $A = \frac{(s'+s''+\ldots)!}{s'!\,s''!\ldots}$, for in repeated differentiations of a product, the combinations of partial derivatives are the same as in the case of powers of a multinomial algebraic expression; as is evident on considering that

$$\frac{d}{dx}u'u''\ldots = \left(\frac{d_{\prime}}{dx}+\frac{d_{\prime\prime}}{dx}+\ldots\right)u'u''. \quad \text{(See Note.)}$$

Similar results hold for $v'v''\ldots.$, with the restriction that no derivative of v' or $v''\ldots$, of not greater order than $t'+t''+\ldots$ is infinite; and the coefficient of the term which does not vanish $= \frac{(t'+t''+\ldots)!}{t'!\,t''!\ldots}$.